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Brief article

Free versus anchored numerical estimation: A unified approach

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ABSTRACT

Children's number-line estimation has produced a lively debate about representational change, supported by apparently incompatible data regarding descriptive adequacy of logarithmic (Opfer, Siegler, & Young, 2011) and cyclic power models (Slusser, Santiago, & Barth, 2013). To test whether methodological differences might explain discrepant findings, we created a fully crossed 2 × 2 design and assigned 96 children to one of four cells. In the design, we crossed anchoring (free, anchored) and sampling (over-, even-), which were candidate factors to explain discrepant findings. In three conditions (free/over-sampling, free/even-sampling, and anchored/over-sampling), the majority of children provided estimates better fit by the logarithmic than cyclic power function. In the last condition (anchored/even-sampling), the reverse was found. Results suggest that logarithmically-compressed numerical estimates do not depend on sampling, that the fit of cyclic power functions to children's estimates is likely an effect of anchors, and that a mixed log/linear model provides a useful model for both free and anchored numerical estimation.

1. Introduction

In this paper, we attempt to reconcile seemingly incompatible data (Barth & Paladino, 2011; Opfer & Siegler, 2007; Opfer, Siegler, & Young, 2011; Slusser, Santiago, & Barth, 2013) regarding the psychophysical functions that link numbers to children's estimates of numerical magnitude.

The psychophysical functions that link numbers to subjects' estimates of numerical magnitude are both theoretically and practically important. Of theoretical interest, functions generating young children's numerical magnitude estimates have been observed in non-symbolic number discrimination of a wide range of species (for review, see Nieder & Dehaene, 2009), to change abruptly with limited experience (Izard & Dehaene, 2008; Opfer & Siegler, 2007), and to closely track abilities to deal with numbers in other contexts (Booth & Siegler, 2006; Thompson & Siegler, 2010). Thus, just as animals can better discriminate 1 and 10 objects than 101 and 110 objects, so too do children estimate magnitudes of symbols 1 and 10 to differ more than 101 and 110. These results suggest that (1) across development, numerical symbols are linked to an innate "mental number line" that allows infants and other animals to discriminate numbers and match them across modalities (see Fig. 1) and (2) linking between symbolic numbers and mental magnitudes is plastic and undergoes significant change (Opfer & Siegler, 2012).

Psychophysical functions linking numbers and estimates of numerical value have also emerged as practically important. Specifically, functions generating children's numerical estimates correlate highly with real-world behavior, including children's memory for numbers, ability to learn arithmetic facts, math grades in school, and math achievement scores (Booth & Siegler, 2006, 2008; Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Thompson, 2014; Siegler, Thompson, & Schneider, 2011). These findings suggest that children's representations of numerical magnitude play an important role in development of mathematical ability and should be a target for educational interventions.

What psychophysical functions are most likely to generate estimates of numerical value? Across a wide range of tasks and age groups (for review, see Opfer & Siegler, 2012), we have observed two functions as being most likely contenders: the logarithmic function given by Fechner's Law and a standard linear function (see Appendix A)[3]. For example, on number-line estimation tasks,

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[3] Another model of the logarithmic-to-linear shift, suggested by Anobile, Cicchini, and Burr (2012) and Cicchini, Anobile, and Burr (2014), also combines the two equations into a single formula: $y = a\left((1-\lambda)X + \lambda(U/\ln(U))\ln(X)\right)$, where a is a scaling parameter, λ is the logarithmicity of the estimate, and U is the upper bound of the number-line.

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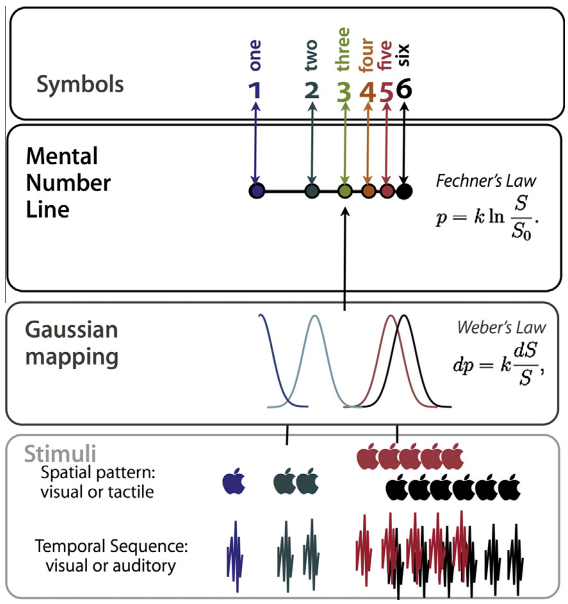

Fig. 1. Model of early numerical magnitude representations (from Opfer & Siegler, 2012).

children are shown a blank line flanked by two numbers (e.g., 0 and 1000) and asked to estimate the position of a third number. Because line length itself is not psychophysically compressive or expansive (Lu & Dosher, 2013), the task provides a relatively straightforward method for assessing compression in numerical magnitude representations.

In many number-line estimation studies, a logarithmic-to-linear shift has been observed. For example, on a 0–1000 task, second graders' median estimates were best fit by a logarithmic function, whereas sixth graders' and adults' median estimates were best fit by the linear function; similarly, over 90% of individual second graders' estimates were better fit by the logarithmic than linear function, whereas the reverse was true of sixth graders and adults (Siegler & Opfer, 2003). This developmental sequence has been observed at different ages with different numerical ranges. It occurs between preschool and kindergarten for the 0–10 range, between kindergarten and second grade for the 0–100 range, between second and fourth grade for the 0–1000 range, and between third and sixth grade for the 0–100,000 range (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Opfer & Siegler, 2007; Siegler & Booth, 2004; Thompson & Opfer, 2010). Similar transitions occur roughly a year later for children with mathematical learning difficulties (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Timing of the changes corresponds to periods when children are gaining extensive exposure to numerical ranges: through counting during preschool for numbers up to 10, through addition and subtraction between kindergarten and second grade for numbers through 100, and through all four arithmetic operations in later elementary school.

Against the idea of a logarithmic-to-linear shift, however, Barth and colleagues (Barth & Paladino, 2011; Slusser et al., 2013) have recently presented evidence that estimates of numerical value may follow cyclic power functions rather than being truly Fechnerian logarithmic functions or arithmetically correct linear functions. For example, on a 0–100 number line task, estimates of 7-year-olds were found to follow a 2-cycle power function originally described by Hollands and Dyre (2000). Indeed, fit of the 2-cycle power function was strongest for 7-year-olds' (R^2 = .968) and 8-year-olds' (R^2 = .995) estimates on the 0–1000 number line task, which we examine in our present study. Further, rather than observing an abrupt, single-trial increase in linearity (as reported in Opfer & Siegler, 2007), Barth and colleagues observed a gradual, age-related increase in value of the exponent of the power function. If true, these quantitative findings are theoretically important. First, they suggest that commonalities between estimates of symbolic and non-symbolic magnitude may be illusory, with estimates of symbolic magnitude being affected by children's prior knowledge of proportions (e.g., 500 is half of 1000). Second, they suggest that changes in numerical magnitude estimates are *quantitative* (in the sense that one parameter in the same function changes over time) rather than *qualitative* (in the sense that different functions are needed to describe younger versus older children's estimates).

1.1. Why different functions? Sampling versus anchoring

To illustrate differences between data cited in support of the logarithmic-to-linear shift account and the proportion-judgment account, it is useful to compare 7- and 8-year-olds' number-line estimates on the 0–1000 task (Fig. 2), where Slusser et al. (2013) found a better fit for the 2-cycle power function over the logarithmic, despite the logarithmic function providing a better fit in data collected by Opfer and Siegler (2007). Given that children's ages and numeric ranges were the same, something must explain these discrepant findings.

One potential cause of the discrepancy is methodological differences in *sampling* (Barth, Slusser, Cohen, & Paladino, 2011; Slusser et al., 2013), with fit of the logarithmic function being an artifact of sparsely sampling at the upper ranges (e.g., obtaining few estimates for numbers 750–1000) and heavily sampling at lower ranges (e.g., obtaining many estimates for numbers 0–250). As Slusser et al. (2013) write, "there is a resounding tendency for researchers to sample heavily from the lower end of the number line and scarcely from the upper end.... This practice focuses on participants' propensity to overestimate small numbers, but yields little data to reveal the details of underestimation patterns for larger numbers" (p. 4). This observation has potential force. As can be seen in Fig. 2, Opfer and Siegler (2007) collected estimates for 13 numbers in the 0–250 range and 3 numbers in the 750–1000 range, whereas Slusser et al. (2013) collected estimates for 7 numbers in each range.

Another potential cause of the discrepancy is methodological differences in *anchoring* (Opfer et al., 2011), with fit of the 2-cycle power function being an effect of experimenters telling children the placement of 500. In the typical number-line task (Siegler & Booth, 2004; Siegler & Opfer, 2003, Exp. 1; Booth & Siegler, 2006, Exp. 2; Laski & Siegler, 2007; Opfer & Martens, 2012; Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008; though see Siegler & Booth, 2004, Exp. 2, and Booth & Siegler, 2006, Exp. 1, for use of anchors), children are given no supervision on any of their number line placements. In contrast, in all studies finding a superior fit of the 2-cycle power function, children's estimate of the halfway point is anchored. For example, in Slusser et al. (2013), children were told, "Because 500 is half of 1000, it goes right in the middle between 0 and 1000. So 500 goes right there, but it's the only number that goes right there." Given previous training studies (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008) finding that a single trial of feedback can increase linearity of estimates, such anchors seem highly likely to affect children's estimates.

While these two potential causes of the discrepancy in findings are not mutually exclusive, each cause has different theoretical implications. From the logarithmic-to-linear-shift account, differences in sampling are predicted to be minor because oversampling has only a small impact on absolute fits and no impact on model selection. In contrast, from the proportion-judgment account, dif-

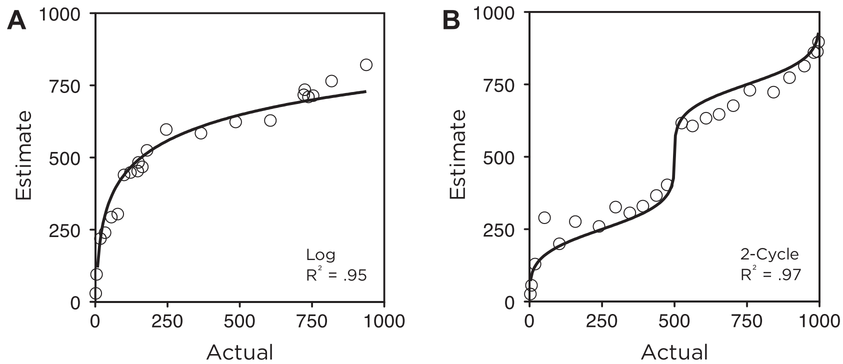

Fig. 2. Discrepant results in 0–1000 number-line estimation: A. Pretest number-line estimates from Opfer & Siegler, 2007; B. Number-line estimates redrawn from Slusser et al. (2013).

ferences in anchoring are predicted to be minor because if children naturally use proportional anchors (e.g., 500 as half of 1000), anchors tell children nothing new. Thus, in addition to helping to explain the discrepancy in previous results, an effect of either sampling or anchoring is meaningful.

1.2. The current study

To test the impact of anchoring and sampling on fit of the logarithmic and 2-cycle power functions to children's number line estimates, we created a fully crossed 2 × 2 design and assigned 96 children to one of four conditions. Two of these conditions were direct replication attempts of previous findings: free/oversampling was a direct replication of Opfer and Siegler (2007), and anchored/even-sampling was a direct replication of Slusser et al. (2013). In these two cells, we expected to replicate previous findings (i.e., best fit by logarithmic function for free/oversampling; best fit by 2-cycle power function for anchored/even-sampling as in Fig. 2). The remaining two conditions had not been tested previously. For the over-sampling/anchored cell, both the logarithmic-to-linear-shift and proportion-judgment accounts expect slightly worse fits of their preferred models (though for different reasons). Thus, the most interesting condition is the free/even-sampling cell. If fit of the logarithmic function is simply an artifact of over-sampling, estimates in this condition are expected to be best fit by a 2-cycle power function. In contrast, if fit of the cyclic power function is an effect of anchoring, estimates are expected to be best fit by a logarithmic function.

2. Method

2.1. Participants

Participants were 96 first and second grade students (M = 7.62 years, SD = 0.59 years; 55% females; 74% Caucasian, 8% Biracial, 6% Asian, 5% African American, 4% Native American, and 2% Hispanic) who attended one of five public elementary schools in Norman, OK. Two female research assistants presented the procedure.

2.2. Procedure and design

All children completed the estimation task one-on-one with a trained experimenter. For each problem, children were shown a 21.8-cm line, with left endpoint labeled 0 and right endpoint labeled 1000. Children's task was to estimate the position of a third number by making a hatch mark on the line.

Children differed in the numbers that they estimated and instructions they received. Specifically, children were randomly assigned to one of four fully-crossed experimental conditions that differed with respect to the numbers they estimated (oversampling/even-sampling conditions) and whether information was given about the location of 500 (anchored/free conditions).

In the oversampling conditions, children were asked to estimate positions of 2, 5, 18, 34, 56, 78, 100, 122, 147, 150, 163, 179, 246, 366, 486, 606, 722, 725, 738, 754, 818, and 938. These numbers had been used in Opfer and Siegler (2007). In the even sampling conditions, children were asked to estimate positions of 3, 7, 19, 52, 103, 158, 240, 297, 346, 391, 438, 475, 525, 562, 609, 654, 703, 760, 842, 897, 948, 981, 993, and 997. These numbers had been used in Slusser et al. (2013).

In the anchored conditions, children received these instructions (adapted from Slusser et al., 2013): "*This task is with number lines. There will be a number up here.* [Researcher pointed to top left corner of blank number line data sheet where the to-be-estimated number was located.] *Your job is to show me where that number goes on a number line like this one. Each number line will have a 0 at this end* [Researcher pointed to 0.] *and 1000 at the other end* [Researcher pointed to 1000.]. *When you decide where the right place for the number is, I want you to make a mark through the line like this.* [Researcher made a vertical hatch mark in the air in front of child.] *Can you show me where 0 goes? Great! Now, can you show me where 1000 goes?* [Researcher provided corrective feedback if participant did not mark right location for these two numbers.] *So if this is 0, and this is 1000, where would you put 500?* [Researcher provided corrective feedback on location of 500.] *Because 500 is half of 1000, it goes right in the middle between 0 and 1000. So 500 goes right there* [Researcher pointed to vertical hatch mark that indicated correct location of 500], *but it's the only number that goes there. I am going to show you a lot of numbers, so just mark where you think each one should go. Don't spend too long thinking about each one. I will read you the number above the line, and then you should decide where that number goes. Are you ready to give it a try?*"

Children assigned to the free conditions received the same instructions except these children were not asked to estimate 500, and thus were not given a midpoint anchor to guide their estimates.

3. Results

To ensure that our random assignment to condition resulted in equivalent groups, we first confirmed that age did not differ significantly by experimental condition, $F(1,92) = .41$, $p > .05$.

3.1. Comparison of logarithmic and 2-cycle power models

Following Slusser et al. (2013)'s analyses of the 0–1000 number-line tasks, we calculated the fit of the logarithmic and 2-cycle model power models to median estimates, compared them using AICc, and we calculated the proportion of individual children who were best fit by the logarithmic model using AICc and BIC. As illustrated in Table 1, the results from the free/over-sampled condition replicated those found in Opfer and Siegler (2007), with the logarithmic model being the more likely data-generating model (>99.99%) for median estimates and providing a better fit for 83% of children's estimation patterns. Results from the anchored/even-sampled condition also replicated those found by Slusser et al. (2013), with the 2-cycle power model being the more likely data-generating model (99%) for median estimates and providing a better fit to 71% of participants' estimation patterns. Thus, our current findings replicate the discrepant ones reported in Opfer and Siegler (2007) and Slusser et al. (2013).

Results from the remaining two conditions were critical for revealing the source of discrepancy between the free/oversampling and anchored/even-sampling conditions. As expected, when children were tested with anchors and oversampling, both the absolute R^2 of the logarithmic and 2-cycle power models was reduced, as well as the relative differences. This suggests either that oversampling penalizes the power function (Slusser et al., 2013), that anchors increase linearity of estimates (Opfer & Thompson, 2008; Thompson & Opfer, 2008), or both. The last condition – presenting children with no anchors and even sampling – addresses these possibilities, with the result that the 2-cycle power model recovered somewhat in overall fit but with logarithmic model still being more likely (98.7%) to be the data-generating function for median estimates and 67% of individual children's estimates being best fit by the logarithmic model. Thus, rather than the fit of the logarithmic model being an artifact of the numbers tested, numerical estimates – when free of anchors – appear to follow a logarithmic function.

3.2. Toward a unified approach for free and anchored numerical estimation

Why might a midpoint anchor raise the relative fit of the 2-cycle power function to be equal or greater than the logarithmic function? One possibility, suggested by Opfer and Siegler (2007), is that a midpoint anchor causes 7-year-olds to improve their estimates overall (thereby increasing fit of the linear function) because the position of 500 on a 0–1000 number line is like the position of 50 on a 0–100 number line, where 7-year-olds already place numbers linearly. An interesting implication of this idea is that both free and anchored numerical estimation might be predicted using the mixed log-linear model suggested by Anobile et al. (2012) for estimates of non-symbolic number, which predicts estimates as the sum of weighted logarithmic and linear components (see Appendix A). If this view were correct, the weight of the logarithmic component (λ) would be expected to be lower in the anchored conditions than the free conditions.

Another possibility is that distinct estimation strategies are used depending on age and the provision of anchors. For example, following Hollands and Dyre's (2000) account, the 1-cycle and 2-cycle power models are appropriate when subjects use the strategy of comparing a number either to both end-points alone (1-cycle) or with the addition of a central reference anchor (2-cycle). To detect the use of a mix (any mix, from 0% to 100% 1-cycle) of these two strategies, Hollands and Dyre (2000) proposed a mixed cyclic power model. Additionally, both Slusser et al. (2013) and Hollands and Dyre (2000) hypothesized the strategy of using only a single endpoint, in which case a 0-cycle power model would be predicted to fit data, and thereby require an extension of the mixed 1-cycle and 2-cycle model. Hollands and Dyre themselves foresaw the need for this adaptation, and they described a procedure for extending their model, which we followed to include a component for a 0-cycle power model (Appendix A). Thus, if this view were correct, the weights of the 0-cycle component (w_1) and 1-cycle component (w_2) would be expected to be lower in anchored conditions than the free conditions, whereas the weight of the 2-cycle component ($1 - [w_1 + w_2]$) would be greater in the anchored conditions than the free conditions.

Table 1
Statistics from median group estimates and percent of participants who were best fit by the log model.

	Statistics from median estimates				% participants best fit by the log model	
	R^2		ΔAICc	p (log)	AICc	BIC
	Log	2-cycle				
Free						
Over-sampled	.95	.00	−62.89	>99.99	83.33	83.33
Even-sampled	.83	.72	−8.73	98.74	66.67	66.67
Anchored						
Over-sampled	.83	.81	.31	46.10	58.33	58.33
Even-sampled	.70	.93	35.84	.01	29.17	29.17

By comparing the fit of the mixed log-linear and mixed cyclic power models to data in our four conditions, we were thereby able to test for multiple strategies of numerical estimation and to test rival explanations for the effect of anchors. Table 2 provides the parameter estimates for each model across conditions. Table 3 provides details on the model comparison statistics.

As illustrated in Table 3 and Fig. 3, the mixed cyclic power model (MCPM) appears much less suited for providing a unified model of numerical estimation than the mixed log-linear model (MLLM). First, despite being simpler, the MLLM had very high fits to median estimates across all four conditions (R^2s .95–.97), and provided the best fitting function to the majority of individual subjects across all four conditions as well. In contrast, the 6-parameter MCPM had very high fits for three conditions, but it had a relatively poor fit in the free/oversampled condition (R^2 = .64).

More importantly, the MCPM appears to overfit data much more than the MLLM. This is most evident in the highly variable parameter estimates in Table 2. For example, the β-parameter of the MCPM (like the λ-parameter of the MLLM) is thought to index the degree of compression or expansion in the mental scaling of numeric magnitude. However, the estimated β-values associated with the one- and two-cycle components ranged roughly 30-fold across conditions (from a highly compressive .33 to a highly expansive 9.66), varied as greatly among subjects within the same condition, and with no discernible pattern. Indeed, even the β-value associated with the zero-cycle power component (which is quite similar to the mixed log-linear model) did not correlate with age or condition. This is concerning because it indicates that the parameters of the model do not have a clear psychological meaning, and it cautions against overinterpretation of the β-value as a marker of developmental changes in numerical magnitude representations. In contrast, the λ-parameter of the MLLM showed much less variability among subjects, and a systematic decrease (from logarithmic to linear) as a function of age ($r(94) = -.34$, $p < .001$) and anchors, $t(94) = 2.39$, $p < .05$, $d = .15$ (Fig. 3). Moreover, results remained much the same even after the number of free parameters in the MCPM was reduced to 4 by making the three sub-power models share the exponent β (see Appendix A). Thus, though the MCPM was indeed favored for median estimates (but not the majority of individual estimates) in one of the four conditions, it is difficult to see how the model parameters map onto a psychological reality that can unify findings regarding diverse forms of numerical estimation or provide a coherent depiction of numerical magnitude representations.

Table 2
Mean estimates (and SDs) for parameter values from the mixed log-linear model (MLLM) and mixed cyclic power model (MCPM).

	MLLM		MCPM						
	a	λ	α	w_1	β_1	w_2	β_2	$1-(w_1+w_2)$	β_3
Free									
Over-sampled	.81 (.15)	.75 (.34)	−140.52 (353.62)	.03 (.07)	.53 (.57)	.82 (.28)	6.09 (22.04)	.15 (.27)	3.10 (11.75)
Even-sampled	.81 (.10)	.55 (.36)	−249.08 (442.97)	.06 (.21)	.37 (.28)	.72 (.35)	4.79 (20.89)	.21 (.30)	6.22 (21.48)
Anchored									
Over-sampled	.80 (.09)	.59 (.31)	−83.35 (352.81)	.01 (.02)	.33 (.31)	.72 (.29)	1.43 (5.38)	.28 (.29)	4.66 (21.10)
Even-sampled	.80 (.11)	.37 (.33)	−245.85 (427.65)	.12 (.20)	.58 (.64)	.40 (.29)	9.66 (28.74)	.48 (.33)	.47 (.40)

Table 3
Statistics from median group estimates and percent of participants who were best fit by the mixed log-linear model (MLLM).

	Statistics from median estimates				% participants best fit by the MLLM	
	R^2		ΔAICc	p(MLLM)	AICc	BIC
	MLLM	MCPM				
Free						
Over-sampled	.95	.64	−55.22	99.99	100.00	100.00
Even-sampled	.97	.96	−17.93	99.99	100.00	100.00
Anchored						
Over-sampled	.96	.96	−16.92	99.98	91.67	91.67
Even-sampled	.95	.99	15.65	.00	66.67	66.67

4. Discussion

In this paper, we sought to reconcile seemingly incompatible data (Barth & Paladino, 2011; Opfer & Siegler, 2007; Opfer et al., 2011; Slusser et al., 2013) regarding the psychophysical functions that link numbers to children's estimates of numerical magnitude. Specifically, we sought to identify influences of sampling and anchoring on the absolute and relative fits of power and logarithmic functions. Additionally, we sought to test whether a mixed log-linear model suggested by Anobile et al. (2012) provided a useful framework for understanding both free and anchored numerical estimation.

The results of our study indicate that young children's free estimates of numerical magnitude do tend to increase logarithmically with actual value. This finding held regardless of whether numbers

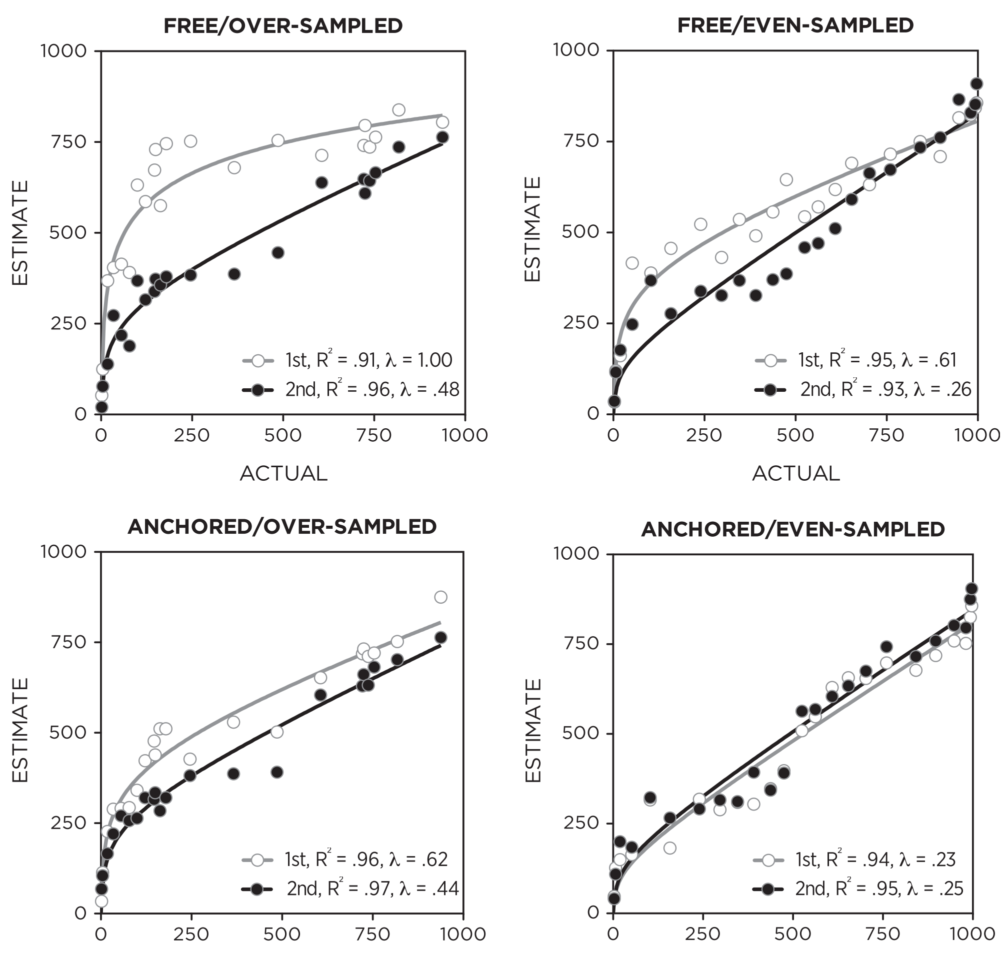

Fig. 3. Median estimates by age group (1st vs. 2nd grade) and experimental condition. A mixed log-linear regression model provided an excellent fit across conditions, with the degree of logarithmicity (λ) being reduced by age group, even-sampling, and anchors.

that were presented to children oversampled the low end of the range or sampled all numbers equally. This result is not consistent with the speculation of Barth et al. (2011) and Slusser et al. (2013) that superiority of fit of the logarithmic function to the cyclic power function is an artifact of sampling. This was an important issue to test because the only previous study examining the relative fits of the two models to free estimates (Opfer et al., 2011) had relied on data that used over-sampling and found that over 90% of 576 individual children's estimates were fit by the linear and logarithmic functions better than cyclic power functions. Thus, we can be confident that a logarithmic-to-linear shift exists in free numerical estimation.

Results also suggest that it is not very likely that young children spontaneously make use of numerical proportions when estimating positions of numbers on number lines. This is a key claim of the proportion-judgment account of numerical estimation, and it guides the choice of models for testing. Against this view, however, few second graders know that 500 is half of 1000. Thus, telling them this fact in the context of number-line estimation is likely to have a large effect on their estimates. Consistent with this idea, only 33% of children making *free* estimates (with even sampling) were best fit by the two-cycle power function, whereas 70% of children making *anchored* estimates (with even sampling) were best fit by the cyclic power function (Table 1). This result would not be expected if children already knew the proportions being given by Slusser et al. (2013) in their instructions to children.

As a framework for modeling both free and anchored number-line estimates, the mixed log-linear model suggested by Anobile et al. (2012) proved useful. Indeed, when we compared this mixed log-linear model to a mixed cyclic power model that incorporates a simple power function, a one-cycle power function, and a two-cycle power function, the simpler log-linear model best fit 100% of children's free estimates and 67–92% of children's anchored estimates (Table 3). Additionally, the parameter values of the mixed log-linear model corresponded to the predicted effects of age and condition (i.e., lower λ-values with age and anchors), whereas the β-values of the mixed cyclic power model varied wildly and with no discernable pattern. These results suggest that the λ-value of a mixed log-linear model is more appropriate for gaining insights into developmental changes in the mental scaling of numeric magnitude than are the β-values of the mixed cyclic power function.

Beyond these issues of modeling, however, results suggest that anchors have a powerful effect on children's number-line estimates. We found that the largest impact of anchors was to increase linearity of estimates (regardless of sampling), not to cause children's estimates to follow some mixture of power functions. This result is consistent with a number of training studies of children's number-line estimates (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008), where anchors were chosen strategically to highlight how far logarithmic number line placements depart from accuracy. Indeed, the importance of anchors – and midpoint anchors in particular (Parducci, 1968) – for estimation is a well-established phenomenon (Tversky & Kahneman, 1974). An important conclusion from these studies and the present study is that estimates of numerical magnitude are plastic and modifiable by experience. Given that linearity of children's numerical estimates correlates highly with real-world behavior, including children's memory for numbers, their ability to learn arithmetic facts, their math grades in school, and their math achievement scores (Booth & Siegler, 2006, 2008; Fazio et al., 2014; Siegler & Thompson, 2014; Siegler et al., 2011), the present results suggest that providing anchors for numerical magnitude judgments could have an important effect on children's general math proficiency as well.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.cognition.2015.11.015.

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