The Logarithmic-To-Linear Shift: One Learning Sequence, Many Tasks, Many Time Scales

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ABSTRACT—The relation between short-term and long-term change (also known as learning and development) has been of great interest throughout the history of developmental psychology. Werner and Vygotsky believed that the two involved basically similar progressions of qualitatively distinct knowledge states; behaviorists such as Kendler and Kendler believed that the two involved similar patterns of continuous growth; Piaget believed that the two were basically dissimilar, with only development involving qualitative reorganization of existing knowledge and acquisition of new cognitive structures. This article examines the viability of these three accounts in accounting for the development of numerical representations. A review of this literature indicated that Werner's and Vygotsky's position (and that of modern dynamic systems and information processing theorists) provided the most accurate account of the data. In particular, both changes over periods of years and changes within a single experimental session indicated that children progress from logarithmic to linear representations of numerical magnitudes, at times showing abrupt changes across a large range of numbers. The pattern occurs with representations of whole number magnitudes at different ages for different numerical ranges; thus, children progress from logarithmic to linear representations of the 0–100 range between kindergarten and second grade, whereas they make the same transition in the 0–1,000 range between second and fourth grade. Similar changes are seen on tasks involving fractions; these changes yield the paradoxical finding that young children at times estimate fractional magnitudes more accurately than adults do. Several different educational interventions based on this analysis of changes in numerical representations have yielded promising results.

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INTRODUCTION

The relation between short-term and long-term change is among the enduring issues in developmental psychology, one that plays a prominent role in both classical and contemporary theories of cognitive development. Werner (1948, 1957) and Vygotsky (1934/1962, 1998) viewed short-term change as a speeded-up version of long-term change; that is, they believed that the two involve similar qualitatively distinct understandings, that the understandings emerge in the same order, and that they reflect the same underlying processes. Learning theorists such as Kendler and Kendler (1962) also viewed the two as fundamentally similar but, unlike Werner and Vygotsky, viewed both as proceeding through gradual incremental processes with no qualitatively distinct knowledge states. Piaget (1964, 1970) expressed a third perspective; he viewed the two types of change, which he referred to as learning and development, as fundamentally dissimilar. In his view, development creates new cognitive structures; learning merely fills in specific content. The relation of short-term to long-term change continues to be of central interest within contemporary theories: dynamic systems (Fischer & Biddell, 1998; Thelen & Smith, 1998), neo-Piagetian (Case, 1998; Karmiloff-Smith, 1992), and information processing (McClelland, 1995; Siegler & Crowley, 1991).

In this article, we review empirical evidence regarding one learning sequence—the logarithmic-to-linear shift in representations of numerical magnitude. Consistent with the stances of Werner, Vygotsky, and contemporary dynamic systems and information processing theorists, we have observed this developmental sequence over multiple time scales, tasks, individuals, and age groups. In this article, we briefly describe findings regarding this developmental sequence and then discuss some educational implications of our analysis.

THE LOGARITHMIC-TO-LINEAR SHIFT IN CHILDREN'S REPRESENTATIONS OF NUMERICAL MAGNITUDE

The learning sequence that is the focus of this article involves representations of numerical magnitudes. Such representations are central to understanding the meaning of number symbols (e.g., knowing that "6" denotes six objects), to comparing the magnitudes of numbers (e.g., knowing that six is more than four), and to estimating quantities (e.g., knowing whether there are 6, 60, or 600 candies in a jar). Development of numerical magnitude representations is an important educational problem, because the process causes many students difficulty and because immature numerical magnitude representations hinder these students' learning of mathematics.

Numerical estimation tasks have proved particularly useful for providing insights into numerical magnitude representations. As noted in Siegler and Booth's (2005) review of the estimation literature, numerical estimation is a process of translating between alternative quantitative representations, at least one of which is inexact and at least one of which is numerical. For example, number-line estimation requires translating a number into a spatial position on a number line or translating a spatial position on a number line into a number. Similarly, numerosity estimation requires translating a non-numerical quantitative representation (e.g., a picture of marbles in a jar) into a number, and computational estimation involves translating from an exact numerical representation (e.g., 75×29) to an inexact one (about 2,200).

This perspective on estimation closely resembles our perspective on representations of numerical magnitudes. Functions for translating between external symbolic representations of numbers (e.g., numerals) and internal, analog, magnitude representations seem to be central to numerical representation, similarly to estimation. Consistent with this view, numerical and spatial representations activate highly overlapping areas of the horizontal intraparietal sulcus (Ansari, 2008). Given this parallel, it is not surprising that estimation tasks have proved useful for examining numerical magnitude representations.

In addition to providing a window on numerical representations, numerical estimation is an important process in its own right. It is a central part of mathematical understanding, requiring integration of conceptual and procedural knowledge of numbers. Perhaps, for this reason, individual differences in estimation proficiency are strongly related to general measures of mathematical proficiency, such as achievement test scores, and to measures of specific numerical processes, such as arithmetic, numerical categorization, and numerical magnitude comparison (Booth & Siegler, 2008; Dowker, 2003; Laski & Siegler, 2007). Moreover, early estimation skill predicts later success in mathematics (Chard et al., 2005; Jordan, Kaplan, Olah, & Locuniak, 2006).

Numerous recent studies using estimation tasks and other tasks that require translation of numerical stimuli indicate that children consistently progress from logarithmic representations of numerical magnitudes to linear representations. This is evident in whole number estimation, fraction estimation, and whole number categorization. It is also evident in both the long-term changes seen in studies of children of different ages and in the short-term changes seen in studies that include experimental manipulations designed to improve representations of numerical magnitude. Below, we describe some of the evidence for the logarithmic-to-linear sequence with varied time scales, tasks, and age groups.

Long-Term Changes in Estimation of Whole Number Magnitudes

To examine the development of representations of whole number magnitudes, Siegler and Opfer (2003) formulated the number-line estimation task. Participants were shown a blank line flanked by a number at each end (e.g., 0 and 1,000) and asked where a third number (e.g., 150) would fall on the line. This estimation task is particularly revealing about representations of numerical magnitude because it transparently reflects the ratio characteristics of the number system. Just as 150 is twice as large as 75, the distance of the estimated position of 150 from 0 should be twice as great as the distance of the estimated position of 75 from 0. More generally, estimated magnitude (y) should increase linearly with actual magnitude (x), with a slope of 1.00, as in the equation $y = x$.

Across a number of cross-sectional studies using this number-line estimation task (Booth & Siegler, 2006; Laski & Siegler, 2007; Opfer & Siegler, 2007; Siegler & Booth, 2004; Siegler & Opfer, 2003; Thompson & Opfer, 2008; n.d.), a systematic difference between younger and older children's estimates has been evident: Younger children's estimates of numerical magnitude typically follow Fechner's law ($y = k \times \ln x$) and increase logarithmically with actual value (Figure 1). In contrast, older children's estimates increase linearly with actual value.

This developmental sequence is seen at different ages with different numerical scales (Figure 1). It occurs between kindergarten and second grade with the 0–100 scale, between second and fourth grade with the 0–1,000 scale, and between third and sixth grade with the 0–100,000 scale (Opfer & Siegler, 2007; Thompson & Opfer, n.d.; Siegler & Booth, 2004). Thus, as shown in Figure 1A, on the 0–100 number-line estimation task, the best-fitting logarithmic function fit kindergartners' estimates better than did the best-fitting linear function ($R^2 = .75$ vs. $R^2 = .49$). In contrast, on the same task, the best-fitting linear function fit second-graders' estimates better than did the best-fitting logarithmic function ($R^2 = .95$ vs. $R^2 = .88$). This pattern of age-related change is not limited to number-line estimation. Children undergo

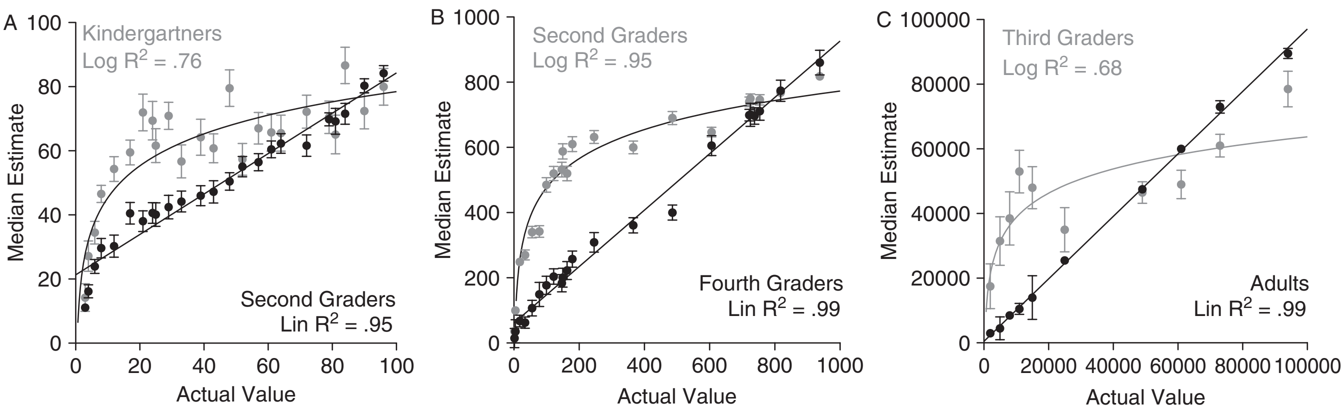

Fig. 1. Long-term changes in estimation of whole number magnitudes. (A) On 0–100 number lines, kindergartners' estimates were better fit by the logarithmic function than by the linear, whereas second-graders' estimates were better fit by the linear function than by the logarithmic (Siegler & Booth, 2004); (B) On 0–1,000 number lines, second-graders' estimates were better fit by the logarithmic function than by the linear, whereas fourth-graders' estimates were better fit by the linear function than by the logarithmic (Opfer & Siegler, 2007); (C) On 0–100,000 number lines, third-graders' estimates were better fit by the logarithmic function than by the linear, whereas adults' estimates were better fit by the linear function than by the logarithmic (Thompson & Opfer, n.d.).

parallel changes from logarithmic to linear representations on numerosity estimation tasks (e.g., being shown jars with 1 and 1,000 marbles on a computer screen and then being asked to hold down a mouse to generate approximately *N* marbles in a previously empty jar). Similar changes with age and experience are seen with measurement estimation (being shown lines of 1 and 1,000 zips and being asked to draw a line of approximately *N* zips). Consistent with the view that these tasks reflect an underlying numerical magnitude representation, strong correlations are present in the variance in individual participants' estimates accounted for by the best-fitting linear function on the three tasks (Booth & Siegler, 2006). Similarly, consistent correlations are present between individual participants' linearity of number-line estimation and their numerical magnitude comparison performance (Laski & Siegler, 2007). Consistent with the view that performance on these estimation tasks reflects a process of educational importance, performance on all three tasks correlates strongly with kindergartners' through fourth-graders' performance on standardized mathematics achievement tests (Booth & Siegler, 2006, 2008; Siegler & Booth, 2004).

Short-Term Changes in Estimation of Whole Number Magnitudes

To investigate whether these long-term changes are part of a more general learning sequence, we conducted three studies of short-term changes in whole number estimation (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008). In these studies, all of which used the number-line estimation task, we used an experimental manipulation to help students who initially used logarithmic representations to adopt a linear representation. Specifically, we provided second-grade students with corrective feedback on their estimates of numbers around 150, the point where the logarithmic and linear functions that pass through 0 and 1,000 are maximally discrepant. After receiving corrective feedback on their estimates for the single number 150, second-graders in all three studies provided estimates that increased linearly with actual value (Figure 2A). This representational change was evident in differences in the median estimates of the treatment groups on pretest and posttest, with pretest estimates being best fit by logarithmic functions (R^2s ranging from .92 to .95) and posttest estimates being best fit by linear functions (R^2s ranging from .91 to .96). As is evident in these figures, second-graders' newly adopted linear representation spanned the entire 0–1,000 range and was not simply a "local fix" to the numbers near 150. This effect across the entire numerical range was consistent with the view that the numerical magnitude representation has psychological reality as a coherent unit, rather than simply being a convenient way of summarizing data.

How quickly did children pass through the logarithmic-to-linear learning sequence? The experimental design of the three studies addressed this question by assessing estimates as children received feedback from trial block to trial block, thereby allowing us to examine how suddenly cognitive change occurred. We found that the shift from the logarithmic representation to the linear one occurred quite abruptly, much more abruptly than what had been observed in studies of transitions in related areas such as addition and numerical insight problems (e.g., Siegler & Jenkins, 1989; Siegler & Stern, 1998). This abruptness of change is indicated by examination of each child's performance before and after the first trial block on which the linear function fit the child's estimates better than the logarithmic function did. Almost no change in the fit of linear function occurred in the trial blocks before it was the

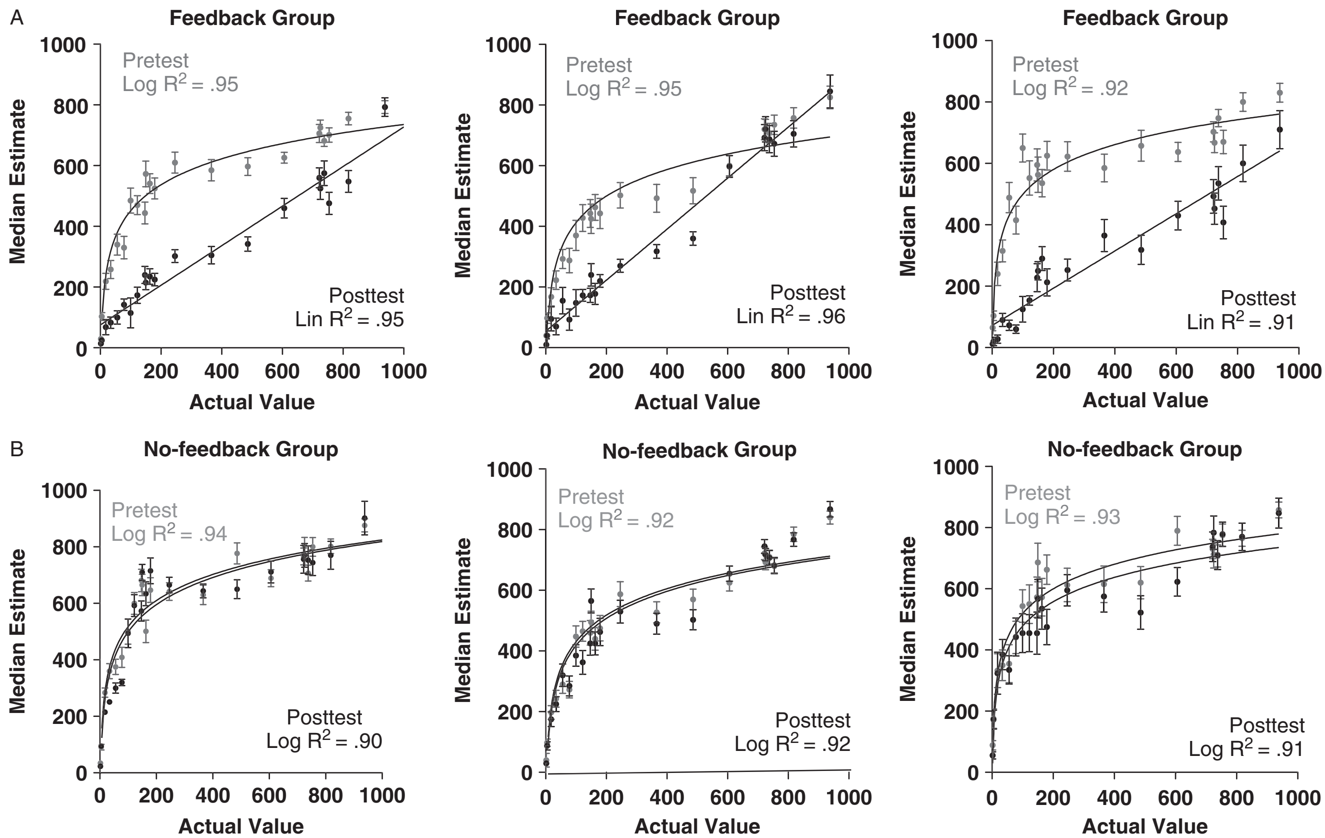

Fig. 2. Short-term changes in estimation of whole number magnitudes. (A) Across three studies (L–R: Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008), logarithmic-to-linear shifts in estimation occurred from pretest (gray circles) to posttest (black circles) when children were given feedback. (B) Across the same studies, very little change in logarithmic estimation patterns occurred when children were not given feedback.

best-fitting function for a given child, and almost no change in the fit of the linear function occurred after that (Figure 3). Rather, the change in the fit of the linear function occurred in just one trial block. Further, without such feedback, the estimation patterns of children in the control groups were nearly identical from pretest to posttest (Figure 2B). Thus, although the cognitive changes occurred quickly and within a single experimental session in these three studies, the learning sequence was identical to the long-term changes seen in cross-sectional studies.

Long-Term Changes in Estimation of Fraction Magnitudes

Estimating the magnitude of fractions on a number line provided a particularly interesting case for the hypothesized sequence of numerical representations. In whole number estimation, adoption of a linear representation consistently increases accuracy on the number-line task. In contrast, applying a linear representation to estimating fractional magnitudes results in *less* accurate performance than applying a logarithmic representation. Thus, increasing use of linear representations of fractional magnitudes cannot be ascribed to general increases in numerical proficiency, which presumably would lead to increasingly accurate performance in estimating the magnitudes of fractions as well as those of whole numbers.

Opfer and DeVries (2008) presented adults with a number line with 1/1,440 at one end and 1/1 at the other; the task was to estimate the positions on the line of other fractions. The task was presented in the context of questions about people's salaries. Participants were told that some people received $1/min and others received $1/1,440 min, and that they should estimate on a number line the location of other salaries relative to these two extremes.

The adults' estimates proved to be a linear function of the size of the denominator ($R^2 = .94$). This meant, for example, that the adults consistently placed 1/60 closer to 1/1 than to 1/1,440. This sounds reasonable until we translate the fractions into decimal equivalents, at which point it becomes clear that this is akin to claiming that .02 is closer to 1 than to 0. Second-graders' estimates on this task also were a function of denominator size, but the best-fitting logarithmic function accounted for far more variance in the second-graders'

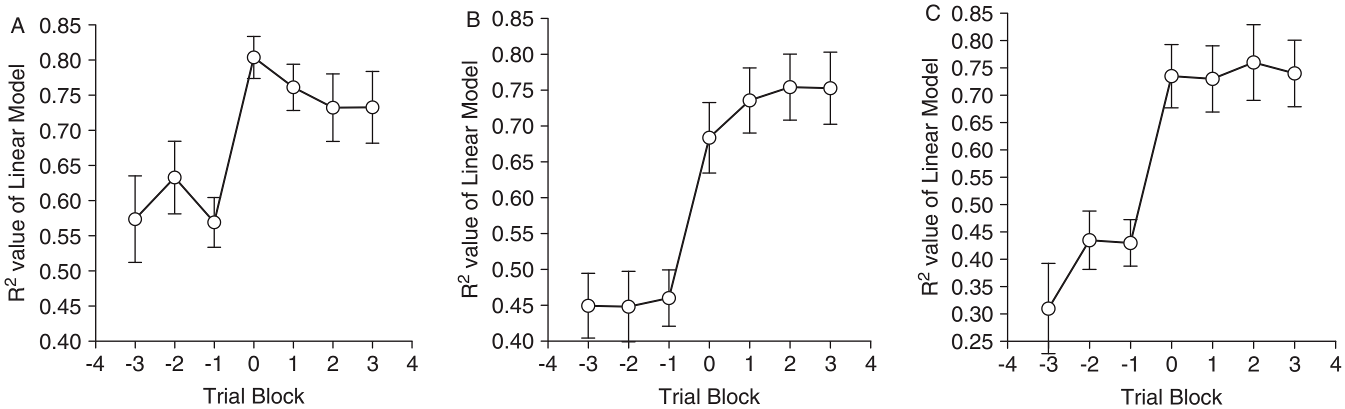

Fig. 3. Trial block to trial block changes in linearity of whole number estimation: Data from (A) Opfer and Siegler (2007), (B) Opfer and Thompson (2008), and (C) Thompson and Opfer (2008). The 0 trial block is the block on which the linear function first provided a better fit to each child's estimates; the −1 trial block is the block before that, and so on.

estimates than the best-fitting linear function ($R^2 = .99$ vs. .75). This led to the seemingly paradoxical finding that the second-graders' estimates were much more accurate than those of adults (Figure 4A). The reason is that when the numerator is held constant, as it was on this task, the relation of denominator size to the fraction's magnitude resembles a logarithmic function much more closely than a linear one. For example, the natural logarithm of 60 is much closer to that of 1,440 than it is to the natural log of 1, just as 1/60 is much closer to 1/1,440 than it is to 1/1.

Two nonintuitive implications of this analysis are that accuracy of fraction number line estimation should decrease with age and that individual children's accuracy of whole number estimation should be negatively related to the accuracy of fraction estimation. The reason for both predictions is that increasing reliance on linear representations improves whole number estimates but harms fractional estimates. To test these hypotheses, Thompson and Opfer (2008) presented 7- to 9-year-olds with the fraction number-line estimation task used by Opfer and DeVries (2008) and also a 0–1,000 whole number estimation task. On the whole number estimation task, the logarithmic-to-linear shift was readily apparent. Moreover, as children learned to provide linear estimates for whole numbers, they seemed to spontaneously apply the same function to fractions (Figure 4B)—an impressive feat of transfer, but one with disastrous results for the accuracy of the estimates of fraction magnitudes. Specifically, the relation between accuracy on the two estimation tasks was very strong and negative ($r = -.80$), with the *accuracy* of children's whole number magnitude estimation accounting

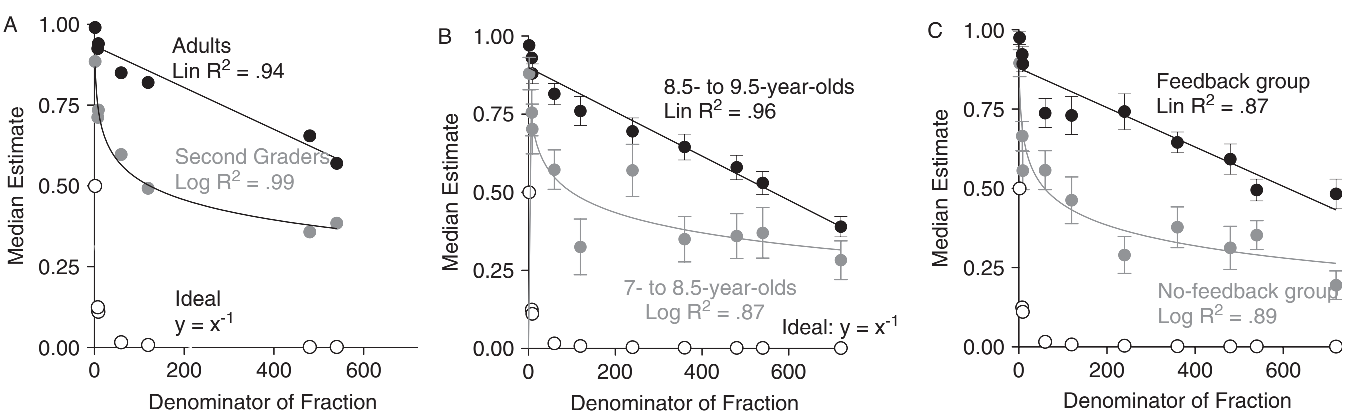

Fig. 4. Long-term and short-term changes in estimation of fraction magnitudes. (A) In Opfer and DeVries (2008), estimates of fraction magnitudes would ideally be a power function of the denominator (white circles). Second-graders' estimates (gray circles) came closer to this ideal than did adults' estimates (black circles). (B) In Thompson and Opfer (2008), younger children's estimates (gray circles) also came closer to the ideal pattern (white circles) than did older children's estimates (black circles). (C) In Thompson and Opfer (2008), children who received feedback on number line problems with whole numbers (black circles) generated estimates that were less accurate than those of children who did not receive feedback (gray circles).

for 64% of the variance in the *inaccuracy* of their fraction magnitude estimation. Also as predicted, the accuracy of fraction magnitude estimation decreased with age.

Short-Term Changes in Estimation of Fraction Magnitudes

To determine whether this decline with age in the accuracy of children's fractional magnitude estimates was part of a more general change in numerical representations, we next examined short-term changes in fractional magnitude estimates after children were given corrective feedback on their number-line estimates with whole numbers. Consistent with the hypothesis, providing feedback on 7- and 8-year-olds' 0–1,000 number-line estimates with whole numbers reduced their accuracy on the fraction estimation task (Figure 4C). The children who received feedback on the whole number estimation task became more linear and more accurate on that task, but their accuracy on the fractions task fell below that of children who received no feedback on either task. Thus, even in a context where the logarithmic-to-linear shift had significant costs, the usual learning sequence was present.

Long-Term Changes in Whole Number Categorization

We also have examined whether the logarithmic-to-linear learning sequence in representations of numerical magnitude is evident in children's categorization of numbers. To test whether long-term changes in children's numerical categorization are related to long-term changes in their representations of numerical magnitude, Laski and Siegler (2007) presented 5- to 8-year-olds with a 0–100 number-line estimation task and also a categorization task for numbers in the same range. On the categorization task, children were told that 1 was a "really small number" and that 100 was a "really big number" and then were asked to categorize numbers between 1 and 100 as "really small," "small," "medium," "big," or "really big." Each child's categorization of each number was assigned a numerical value ranging from 1 for the "really small" category to 5 for the "really big" category. Then, the mean value for the categorizations of each number was computed, and the fit of linear and logarithmic functions to the mean categorization scores for the full set of numbers was calculated.

Kindergartners' mean categorizations of the numbers were better fit by the best-fitting logarithmic function than by the best-fitting linear function ($R^2_{\text{log}} = .97$ vs. $R^2_{\text{lin}} = .81$). In contrast, second-graders' mean categorizations were slightly better fit by the best-fitting linear function than by the best-fitting logarithmic function ($R^2_{\text{log}} = .92$ vs. $R^2_{\text{lin}} = .95$). Moreover, the linearity of individual children's estimation and categorization patterns were highly correlated ($r = .82$ for kindergartners; $r = .80$ for second-graders).

Short-Term Changes in Whole Number Categorization

To determine whether this long-term change in children's categorizations of numbers was part of a more general learning sequence in numerical representations, Opfer and Thompson (2008) examined short-term changes in number categorization after first- and second-graders were given corrective feedback on their number-line estimates. First, a pretest was given to examine the relation between numerical value and categorization judgments prior to training. As on the number-line estimation task, children's pretest categorizations were better fit by a logarithmic function (log $R^2 = .95$) than by a linear function (lin $R^2 = .69$). Fully 90% of individual children's category judgments were better fit by the logarithmic function. Indeed, the more linear the increase with numerical size in the children's number-line estimates, the more linear the increase with numerical size was the children's categorization ($r = .52$). Similarly, the better the fit of the best-fitting logarithmic function for number-line estimation, the better the fit of the best-fitting logarithmic function for numerical categorization ($r = .72$).

We next examined whether feedback designed to improve the linearity of number-line estimates transferred to numerical categorization. As expected, the linear function provided a better fit to the mean category judgments for each number of children who received feedback on their number-line judgments (lin $R^2 = .84$) than for those who did not receive such feedback (lin $R^2 = .68$). The same pattern emerged when looking at the proportion of children who were best fit by each function: The categorization patterns of 46% of children's in the treatment group were best fit by the linear function, whereas the same was true of only 21% of children in the control group. Thus, the pattern of change in response to feedback was again away from a logarithmic pattern of estimates and toward a linear one, and the change in numerical representations extended to the categorization task even without any feedback on that task.

To summarize, the logarithmic-to-linear learning sequence that was originally observed in studies of the development of number-line estimation with whole numbers over periods of years is also evident in numerous other contexts: changes on other tasks in whole number magnitude representations over long time periods, changes over short time periods in response to experimental interventions in whole number magnitude representations, changes in fraction magnitude representations over both long and short time periods, and changes in whole number categorization over both long and short time periods. Moreover, feedback on number-line estimates of whole numbers produces increased reliance on the linear representation, both where it is appropriate (numerical categorization) and where it is inappropriate (number-line estimation with fractions). Thus, the same developmental sequence occurs over varying time periods, varying numerical ranges, varying types of numbers, and varying types of tasks.

EDUCATIONAL IMPLICATIONS

By directly comparing long-term to short-term changes in number-line estimation with whole numbers, number-line estimation with fractions, and categorization of whole numbers, we uncovered a stable unit of cognitive change—the logarithmic-to-linear learning sequence. Use of appropriate representations of numerical magnitudes seems to be important for a wide range of educationally important outcomes. For example, the linearity of children's number-line estimates with whole numbers correlates strongly with the accuracy of their magnitude comparisons, their ability to learn solutions to unfamiliar addition problems (Booth & Siegler, 2008), and their overall scores on mathematics achievement tests (Booth & Siegler, 2006, 2008; Siegler & Booth, 2004). Children with mathematical learning disabilities seem especially hindered by the lack of linear representations of numerical magnitude; they often generate logarithmic patterns of estimates for many years beyond the time when other students have adopted linear representations (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008). Linearity of numerical magnitude representations is not the only important influence on the early growth of numerical understanding; efficient identification of numerals also is predictive of acquisition of other numerical skills (Ramani & Siegler, 2008), and other types of numerical knowledge no doubt are also influential. Nonetheless, linear representations of numerical magnitudes seem to play a particularly important role.

What kind of intervention might help children adopt linear representations of numerical magnitudes, and what impact might such an intervention have on students' learning of mathematics? Booth and Siegler (2008) examined this issue with first-graders who at the outset of the study were at varying points in the transition from logarithmic to linear representations of the 0–100 range. At the beginning of the study, the first-graders were presented with a pretest that examined their number-line estimation and their ability to retrieve answers to 13 addition problems, ranging in difficulty from $1 + 4$ and $5 + 4$ to $38 + 39$ and $49 + 43$. Then, children were trained on four 2-digit + 2-digit problems that they had answered incorrectly on the pretest. All children were presented with each of these problems three times, with feedback regarding the correct answer being provided after each presentation.

A randomly chosen half of the children were also presented with analog linear representations of the addends and sum. This manipulation was intended to inculcate a linear representation of the numbers in the addition problems and ideally of numbers in the 0–100 range more generally. Children in this experimental condition saw a number line with 0 at one end and 100 at the other, then saw the first addend represented by a red bar just above the line, then the second addend represented by a blue bar just below the line, and then the sum represented by a purple bar straddling the line. Thus, if the problem was $43 + 49$, the red bar would be 43% of the number line's length, the blue bar 49% of its length, and the purple bar 92% of its length. The logic was that seeing the linear representations of the addends and sums along the number line would allow children to encode the numerical magnitudes more precisely and thus help them retrieve the answers to the problems.

The linearity of representations of numerical magnitudes proved to be correlated with arithmetic learning, predictive of it, and causally related to it. On the pretest, linearity of number-line estimation was correlated with the number of correct answers to the arithmetic problems ($r = .41$). Pretest linearity of number-line estimates also predicted the amount of learning of the trained addition problems, above and beyond the prediction possible from pretest performance on those arithmetic problems or standardized math achievement test performance. Perhaps most striking, presentation of the analog linear representations of the addends and sum along the number line was causally related to arithmetic learning; it increased the number of addition problems correctly recalled on the posttest and also improved the linearity of the children's number-line estimates. Moreover, the effect of the experimental manipulation was even stronger for measures of the closeness of addition errors to the correct sum than for the number of correct sums, supporting the view that activating the linear representation was the means through which the experimental manipulation produced its effect. If this mechanism were not involved, why else would children who were presented the analog representations of the addends and sums increasingly advance incorrect answers that were close to the sum and decreasingly produce answers that were far from it?

Knowledge of the developmental sequence leading to linearity of numerical magnitude representations has also been the basis for educational interventions with low-income preschoolers. These interventions have shown that inculcating a linear representation of numerical magnitudes for the numbers 1–10 improves preschoolers' number-line estimation, magnitude comparison, numeral identification, counting, and ability to learn the answers to novel arithmetic problems (Ramani & Siegler, 2008; Siegler & Ramani, 2008, in press). The gains have been shown to be stable for at least 2 months after the experimental intervention (Ramani & Siegler, 2008). Applying knowledge of other well-established developmental sequences to the challenge of crafting effective instructional procedures provides a promising means for developing additional educationally useful approaches.

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